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DEPARTMENT OF MATHEMATICS
SECOND SEMESTER 2021/2022 EXAMINATION
MATH 208: LINEAR ALGEBRA II.

ANSWER **ANY FOUR** QUESTIONS TIME: **2 HOURS.**

1. (a) If $S=\{v_1,v_2,\ldots\ldots v_n\}$ is a set of vectors in a vector space V, prove that the linear span of S is the smallest subspace of V containing S.

(b) If V is a subspace of $\mathbb{R}^5$ generated by $\{(1,-1,-1,-2,0),(1,-2,-2,0,-3),(1,-1,-2,-2,1)\}$. Find the homogeneous system whose solution space is V.

2. (a) Prove that a vector space V is a direct sum of its subspaces W and Z if and only if (i) $V=W+Z$ and (ii) $W\cap Z=\{0\}$.

(b) W and Z are subspaces of $\mathbb{R}^4$ generated by $\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$ and $\{(1,1,1,0),(-1,1,2,-1),(0,2,3,-1)\}$, respectively, find a basis for $W\cap Z$.

3. Let V be a vector space of finite dimension and let $F:V\to W$ be linear mapping, prove that $\dim(\ker F)+\dim(imF)=\dim V$

4. (a) Find the basis and dimension of the solution space W of a homogeneous system

$$x+2y \quad +3s+t=0$$
$$2x+3y \quad +3s+t=0$$
[illegible]
$$3x+5y \quad 6s+2t=0$$

(b) If $F:\mathbb{R}^4\to\mathbb{R}^4$ is defined by $F(x,y,z,t)=(x+2y+z+3t,\ x-z,\ 2x+2y+3t, 3x+2y+z+3t)$, find the bases and dimensions of **(i)** imF and **(ii)** $kerF$.

5. (a) If $\{u_1,u_2,\ldots,u_n\}$ is a basis of vector space V and $T:V\to V$, a linear operator on V, define a matrix representation of T relative to the basis.

(b) Let $T(x,y,z)=(x-y+z,\ x-z,\ y+z)$ be a linear operator, $\{u_1=(1,1,1),u_2=(0,1,1),u_3=(0,0,1)\}$ be a basis and $v=(1,2,3)$ in $\mathbb{R}^3$. Verify that $[T]_u[v]_u=[T(v)]_u$.

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6. Let $\{u_1=(1,-1,1),u_2=(1,0,0),u_3=(1,-1,0)\}$ and $\{w_1=(1,1,1),w_2=(0,1,1),w_3=(0,0,1)\}$ be any two bases in $\mathbb{R}^3$ and $T(x,y,z)=(x+y+z,x-y+z,x+y-z)$ be an operator on $\mathbb{R}^3$.

Show that $[T]_w=P^{-1}[T]_u P$, where P is a transition matrix from the basis $\{u_i\}$ to $\{w_i\}$

MATH 208: LINEAR ALGEBRA II, 2021/2022 SESSION

(1a) First, we show that $L(S) \neq \emptyset$ and is closed under vector addition and scalar multiplication. Let $u, v \in L(S)$. Then,

$u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ and $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$ such that $\alpha_i, \beta_i \in k$

Then. $u + v = (\alpha_1 + \beta_1)v_1 + (\alpha_2 + \beta_2)v_2 + \cdots + (\alpha_n + \beta_n)v_n \in L(S)$ because $u + v$ is a linear combination of v_i

Also, $\alpha u = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \cdots + \alpha\alpha_n v_n \Rightarrow$

$L(S)$ is a subspace of V,

ii) Suppose W is a subspace of V containing S, then $\alpha_1 v_1, \alpha_2 v_2, \ldots \alpha_n v_n \in W$

$\Rightarrow W$ contains all linear combination of vectors in S.

$\Rightarrow L(S) \subset W$ and $L(S)$ is the smallest subspace of V containing S.

(1b) To find the homogeneous system whose solution space is V, we write in matrix form and reduces.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ -1 & -2 & -1 & y \\ -2 & -2 & -2 & z \\ -2 & 0 & -2 & s \\ 0 & 3 & 1 & t \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -2 & -2 & y-x \\ 0 & -1 & -3 & z-x \\ 0 & 2 & 0 & s+2x \\ 0 & 3 & 1 & t \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -1 & -2 & y-x \\ 0 & 0 & -1 & z-y \\ 0 & 0 & -4 & s+2y \\ 0 & 0 & -5 & t+3y-3x \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -1 & -2 & y-x \\ 0 & 0 & -1 & z-y \\ 0 & 0 & 0 & s-4z+2y \\ 0 & 0 & 0 & t-3x-5s-8y \end{array}\right)$$

Hence, the homogeneous system is,

$s - 4z - 2y = 0, \; t - 3x - 5s - 8y = 0$

(2a) Proof: Suppose that $V = W \oplus Z$. Then any $v \in V$ ~~and $v \in W$~~ can uniquely be written as $v = w + z$ such that $w \in W$ and $z \in Z$.

In particular, $V = W + Z$.

Also, suppose $v \in W \cap Z$, then $v = 0 + v$ such that $0 \in W$ and $v \in Z$. If $v = v + 0$ such that $v \in W$ and $0 \in Z \Rightarrow v = 0$

Hence $W \cap Z = \{0\}$.

(2b)
$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 3 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & -3 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, the basis for $W \cap Z = \{(1,1,0,-1), (0,1,3,1), (0,0,1,1)\}$ and the $\dim(W \cap Z) = 3$

③ Proof: Since V is a finite dimension, Let $\dim V = n$ and $\dim(\ker F) = K$.
Suppose $K = n$, then $V = \ker F \Rightarrow F(v) = 0 \ \forall v \in V$. Hence, $\operatorname{Im} F = \{0\}$ and $\dim(\operatorname{im} F) = 0$.
Suppose that $1 \leq K \leq n$, we show that $\dim(\operatorname{im} F) = n - K$.
Let $\{v_1, v_2, \cdots v_k\}$ be a basis for $\ker F$, we can extend this basis to that of V. i.e $S = \{v_1, v_2, \cdots v_k, v_{k+1}, v_{k+2}, \cdots v_n\}$ is a basis for V.
We now prove that $T = \{F(v_{k+1}), F(v_{k+2}), \cdots F(v_n)\}$ is a basis for $\operatorname{Im} F$.
Let $w \in \operatorname{Im} F$, then $w = F(v)$ for some $v \in V$. Since S is a basis for V, we can write $v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots + c_n v_n$ where c_i are scalars.

$$w = F(v) = F(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + c_{k+1} v_{k+1} + \cdots + c_n v_n)$$

$$= c_{k+1} F(v_{k+1}) + c_{k+2} F(v_{k+2}) + \cdots + c_n F(v_n) \text{ since } F(v_j) = 0 \ \forall j = 1, 2, \cdots k$$

Hence, T spans $\operatorname{im} F$ i.e T is linearly independent,
Suppose that

$$c_{k+1} F(v_{k+1}) + c_{k+2} F(v_{k+2}) + \cdots + c_n F(v_n) = 0$$

Then, $F(c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots + c_n v_n) = 0$
Hence, $c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots + c_n v_n \in \ker F$ and can be written as a linear combination of vectors in S.

i.e $c_{k+1} v_{k+1} + c_{k+2} v_{k+2} + \cdots + c_n v_n = d_1 v_1 + d_2 v_2 + \cdots + d_k v_k$

where

$$d_1 v_1 + d_2 v_2 + \cdots + d_k v_k - c_{k+1} v_{k+1} - c_{k+2} v_{k+2} + \cdots + c_n v_n = 0$$

Since S is independent $\Rightarrow d_1 = d_2 = \cdots = d_n = c_{k+1} = c_{k+2} = \cdots = c_n = 0$
$\Rightarrow T$ is linear independent and $\dim(\operatorname{im} F) = 0$,

Hence. $\dim V = \dim(\ker F) + \dim(\operatorname{im} F)$ D,

④a

$$x + 2y + 3s + t = 0$$
$$2x + 3y + 3s + t = 0$$
$$x + y + 2z + 2s + t = 0$$
$$3x + 5y + 6s + 2t = 0.$$

Now, to find the basis and dimension of the homogeneous system, we write the system of eqn. in a matrix form and reduces.

i.e $\begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 3 & 5 & 0 & 6 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & -1 & 0 & -3 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -3 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & -1 & 0 & -3 & -1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Thus, dim W = No's of Unknown − No's of non-zero rows,

$$= 5 - 3 = 2$$

$\Rightarrow \dim W = 2$.

Now, to find the basis i.e

$$2z + 2s + t = 0$$

$$-y - 3s - t = 0$$

$x + 2y + 3s + t = 0$, we use the free variables i.e

Let $s = 1, t = 0$, ~~$z=0 \Rightarrow z = 0, y = 0$,~~

$\Rightarrow z = 1, y = -3, x = 3$

$(3, -3, 1, 1, 0)$

and if $s = 0, t = 1 \Rightarrow z = 1/2, y = -1, x = 1$

$(1, -1, 1/2, 0, 1)$

Hence, Basis $W = \{(3, -3, 1, 1, 0), (1, -1, 1/2, 0, 1)\}$

Note that:

free variables are variables which does not appear in the beginning of any of the equation.

(4b) To find the bases and dimension of Imf, we use the usual basis of $\mathbb{R}^4$ generates the image of $\mathbb{R}^4$.

$$f(x,y,z,t) = (x+2y+z+3t,\ x-z,\ 2x+2y+3t,\ 3x+2y+z+3t).$$

$f(1,0,0,0) =$ ~~$(1,0,0,0)$~~ $(1,1,2,3)$

$f(0,1,0,0) = (2,0,2,2)$, $f(0,0,1,0) = (1,-1,0,1)$

$f(0,0,0,1) = (3,0,3,3)$

In matrix form, we have;

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 0 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & 0 & 3 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & -4 \\ 0 & -2 & -2 & -2 \\ 0 & -3 & -3 & -6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & -4 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, the basis of Imf = 3 and

dim (Imf) = $\{(1,1,2,3), (0,-2,-2,-4), (0,0,0,2)\}$.

to find the basis and dimension of ker F, we have

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 2 & 2 & 0 & 3 & 0 \\ 3 & 2 & 1 & 3 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & -2 & -2 & -3 & 0 \\ 0 & -2 & -2 & -3 & 0 \\ 0 & -4 & -2 & -6 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & -2 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right)$$

$\Rightarrow z = 0, \; -2y - 2z - 3t = 0, \; x + 2y + z + 3t = 0$

Hence, dimension (ker F) = No's of Unknown − No's of Non-zero rows

$= 4 - 3 = 1$

Basis (ker F) $= (2, -3/2, 0, 1)$

if $t = 1, y = -3/2, x = 2$

(5a) The transpose of the below matrix of coefficients denoted by $[T]_s$ is called the matrix representation of T relative to the basis S. Let T be a linear operator (transformation) from a vector space V into itself i.e

$T: V \rightarrow V$ and suppose $S = \{U_1, U_2, U_3 \cdots U_n\}$ is a basis of V then $T(U_1), T(U_2) \cdots T(U_n)$ are vectors in V and so each is a linear combination of the vectors in the basis S.

i.e $T(U_1) = a_{11}U_1 + a_{12}U_2 + \cdots + a_{1n}U_n$

$T(U_2) = a_{21}U_1 + a_{22}U_2 + \cdots + a_{2n}U_n$

$\cdots$

$T(U_n) = a_{n1}U_1 + a_{n2}U_2 + \cdots + a_{nn}U_n$

$\Rightarrow [T]_s = \{[T(U_1)]_s, [T(U_2)]_s, [T(U_3)]_s, \cdots, [T(U_n)]_s\}$.

(5b) $T(x,y,z) = (x-y+z, x-z, y+z)$ and $U = U_1, U_2, U_3 = (1,1,1), (0,1,1), (0,0,1)$

$V = (1,2,3)$

$[T]_u [V]_u = [T(V)]_u$

$\Rightarrow [V]_u =$

$(1,2,3) = xU_1 + yU_2 + zU_3 \Rightarrow (1,2,3) = x(1,1,1) + y(0,1,1) + z(0,0,1)$

$\Rightarrow x = 1, y = 1, z = 1$ and $[V]_u = (1,1,1)$ or $\begin{pmatrix}1\\1\\1\end{pmatrix}$

Now, we find the coordinates of $(a,b,c)\in\mathbb{R}^3$ relative to the basis U.

$(a,b,c) = xU_1 + yU_2 + zU_3 = x(1,1,1) + y(0,1,1) + z(0,0,1) \Rightarrow x=a,\ y=b-a,\ z=c-b$

Thus, $(a,b,c) = aU_1 + (b-a)U_2 + (c-b)U_3$

$T(U_1) = T(1,1,1) = (1,0,2) = U_1 - U_2 + 2U_3$

$T(U_2) = T(0,1,1) = (0,-1,2) = -U_2 + 3U_3$

$T(U_3) = T(0,0,1) = (1,-1,1) = U_1 - 2U_2 + 2U_3$

$$[T]_u = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -2 \\ 2 & 3 & 2 \end{pmatrix} \quad \text{Thus, } [T]_u[V]_u = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -2 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix}.$$

Also,

$T(v) = T(1,2,3) = (2,-2,5) = (a,b,c)$

$[T(v)]_u = 2U_1 - 4U_2 + 7U_3 = \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix}$

Hence;

$$[T]_u[V]_u = \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix} = [T(v)]_u$$

$\Rightarrow [T]_u[V]_u = [T(v)]_u$

Verified.

(6) $U_1, U_2, U_3 = (1,-1,1), (1,0,0), (1,-1,0)$ and $w_1, w_2, w_3 = (1,1,1), (0,1,1), (0,0,1)$

$T(x,y,z) = (x+y+z,\ x-y+z,\ x+y-z)$

$(a,b,c) = xU_1 + yU_2 + zU_3 \Rightarrow x = c,\ y = a+b,\ z = -(b+c)$

$(a,b,c) = cU_1 + (a+b)U_2 - (b+c)U_3$ and,

$(a,b,c) = xw_1 + yw_2 + zw_3 \Rightarrow x = a,\ y = b-a,\ z = c-b$

$(a,b,c) = aw_1 + (b-a)w_2 + (c-b)w_3$.

Thus,

$T(w_1) = T(1,1,1) = (3,1,1) = 3w_1 - 2w_2$

$T(w_2) = T(0,1,1) = (2,0,0) = 2w_1 - 2w_2$

$T(w_3) = T(0,0,1) = (1,1,-1) = w_1 - 2w_3$

$$\Rightarrow [T]_w = \begin{pmatrix} 3 & 2 & 1 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Also,

$T(U_1) = T(1,-1,1) = (1,3,-1) = -U_1 + 4U_2 - 2U_3$

$T(U_2) = T(1,0,0) = (1,1,1) = U_1 + 2U_2 - 2U_3$

$T(U_3) = T(1,-1,0) = (0,2,0) = 2U_2 - 2U_3$

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$$[T]_u = \begin{pmatrix} -1 & 1 & 0 \\ 4 & 2 & 2 \\ -2 & -2 & -2 \end{pmatrix}$$

To find P (a transition matrix from the basis $\{U_i\}$ to $\{w_i\}$

$w_1 = (1,1,1) = U_1 + 2U_2 - 2U_3$

$w_2 = (0,1,1) = U_1 + U_2 - 2U_3$

$w_3 = (0,0,1) = U_1 - U_3$

$$\Rightarrow P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -2 & -2 & -1 \end{pmatrix} \quad \Rightarrow P^{-1} \equiv Q = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$$

Thus;

$$P^{-1}[T]_u P = \left[\begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & -2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 4 & 2 & 2 \\ -2 & -2 & -2 \end{pmatrix}\right]\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -2 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -4 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -2 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Hence,

$$[T]_w = \begin{pmatrix} 3 & 2 & 1 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = P^{-1}[T]_u P$$

$$\Rightarrow [T]_w = P^{-1}[T]_u P_{\square}.$$